

ON A LOCAL CHARACTERIZATION OF SOME NEWTON-LIKE METHODS OF R-ORDER AT LEAST THREE UNDER WEAK CONDITIONS IN BANACH SPACES

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ABSTRACT. We present a local convergence analysis of some Newton-like methods of R-order at least three in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first and second Fréchet-derivative of the operator involved. These conditions are weaker than the corresponding ones given by Hernandez, Romero [10] and others [1], [4]-[9] requiring hypotheses up to the third Fréchet derivative. Numerical examples are also provided in this study.

1. Introduction

Many problems in computational sciences and other disciplines are often led to the problem of approximating a solution x^* of the nonlinear equation

$$(1.1) \quad F(x) = 0,$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y .

Newton-like iterative methods [1]-[13] are used to approximate a solution of (1.1) because solutions of these equations can rarely be found in closed form. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence

Received February 12, 2015; Accepted October 26, 2015.

2010 Mathematics Subject Classification: Primary 65D10, 65D99.

Key words and phrases: Newton-like methods, R-order of convergence, Banach space, local convergence, Fréchet-derivative.

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analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semilocal convergence analysis of Newton-like methods such as [1]-[13].

We present a local convergence analysis for the method defined for each $n = 0, 1, 2, \dots$ by

$$(1.2) \quad \begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - \alpha H_n F'(x_n)^{-1}F(x_n), \end{aligned}$$

where x_0 is an initial point, α is a real parameter, $K_n = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$, $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} c_k K_n^k$ and $\{c_k\} \in \mathbb{R}$. Clearly, method (1.2) is well defined, if operator H is well defined. One has that the operator H is,

$$H(K-) : D \xrightarrow{K} L(D, D) \xrightarrow{H} L(D, D),$$

where it is associated to each x_n a polynomial in K_n . That is

$$H_n = H(K_n) = \sum_{k \geq 1} c_k K_n^k,$$

where $c_1 = \frac{1}{2}$. We also denote by K_n^k the composition $K(x)^k = K(x) \circ \dots \circ K(x)$, which is a linear operator in D .

It is worth noticing that if $\alpha = 1$ and $X = Y = \mathbb{R}$ method (1.2) reduces (by choosing H_n appropriately) to well known high convergence order methods for solving equation (1.1). In particular, we have:

- Chebyshev's method [5], [6]: $H_n = \frac{1}{2}K_n$;
- Super-Halley method [8], [2]: $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} \frac{1}{2}K_n^k$;
- Halley method [8], [2]: $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} \frac{1}{2^k}K_n^k$;
- Ostrowski's method [9]: $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} (-1)^k \binom{-\frac{1}{2}}{k} K_n^k$;
- Euler's method [9]: $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} (-1)^k 2^{k+1} \binom{\frac{1}{2}}{k+1} K_n^k$;
- Method (1.2) (for $\alpha = 1$) [10]: $H_n = \frac{1}{2}K_n + \sum_{k \geq 2} c_k K_n^k$, where $\{c_k\}$ is a real decreasing sequence with

$$(1.3) \quad \sum_{k \geq 2} c_k t^k < +\infty \text{ for } |t| < \gamma \text{ for some } \gamma > 0.$$

If one writes method (1.2) in the form (for $\alpha = 1$ and $X = Y = \mathbb{R}$)

$$(1.4) \quad x_{n+1} = G(x_n),$$

where

$$G(t) = t - H(K(t))F'(t)^{-1}F(t),$$

then, method (1.3) has R -order of convergence at least three: Let x^* be a simple root of F and H a function such that $H(0) = 1, H'(0) = \frac{1}{2}$ and $|H''(t)| < \infty$. Then, method (1.4) has R -order of convergence at least three (see, e.g. Gander [9]). A semilocal convergence analysis for method (1.2) in the special case when $\alpha = 1$ and when (1.3) is satisfied was given by Hernandez and Romero in [10]. The semilocal convergence conditions used are (\mathcal{C}) : There exist constants $\beta, \beta_1, \beta_2, \beta_3$ and $x_0 \in D$ such that:

- $(\mathcal{C}_1) \|F'(x_0)^{-1}\| \leq \beta;$
- $(\mathcal{C}_2) \|F'(x_0)^{-1}F(x_0)\| \leq \beta_1;$
- $(\mathcal{C}_3) \|F'(x_0)^{-1}F''(x)\| \leq \beta_2$ for each $x \in D$; and
- $(\mathcal{C}_5) \|F'(x_0)^{-1}(F''(x) - F''(y))\| \leq \beta_3\|x - y\|$ for each $x, y \in D$.

The (\mathcal{C}) conditions were presented [10] in non affine invariant form. However, we present these conditions in affine invariant form in this study. The advantages of results given in invariant form over the results given in non affine invariant form are well known (see, e.g. [2]). Local convergence conditions can be given similarly by simply replacing x_0 by x^* in the (\mathcal{C}) conditions.

However, conditions (\mathcal{C}) for the semilocal or local convergence analysis are very restrictive. As an academic example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, & F'(1) &= 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not satisfy condition (\mathcal{C}_5) . In the present paper we only use hypotheses up to the second Fréchet derivative (see (2.9)–(2.13)). We also avoid condition (1.3). Hence, the applicability of method (1.2) is extended.

The paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Let $U(v, \rho), \bar{U}(v, \rho)$ denote the open and closed balls in X of center v and radius $\rho > 0$.

Let $L_0 > 0, L > 0, M > 0, N > 0$ and $\alpha \in (-\infty, +\infty)$ be given parameters. It is convenient for the local convergence analysis of method (1.2) that follows to introduce functions defined on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} g_1(r) &= \frac{Lr}{2(1 - L_0r)}, \\ g_2(r) &= \frac{MN}{(1 - L_0r)^2}. \end{aligned}$$

Define

$$(2.1) \quad r_A := \frac{2}{L + 2L_0}.$$

Then, it follows from the definition of function g_1 and r_A that $0 \leq g_1(r) < 1$ for each $r \in [0, r_A)$. Let $\{c_k\}$ be a real sequence such that

$$(2.2) \quad \gamma(r) := \frac{MNr}{(1 - L_0r)^2} \lim_{k \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| < 1 \text{ for each } r \in (0, \frac{1}{L_0}).$$

Then, the function

$$(2.3) \quad \varphi(r) = \sum_{k \geq 2} c_k \left(\frac{MN}{(1 - L_0r)^2} \right)^k r^{k-1}$$

is well defined on the interval $(0, \frac{1}{L_0})$. Indeed, we have by the ratio test and (2.2) that

$$\lim_{k \rightarrow \infty} \frac{\frac{(MN)^{k+1} r^k}{(1 - L_0r)^{2(k+1)}} c_{k+1}}{\frac{(MN)^k r^{k-1}}{(1 - L_0r)^{2k}} c_k} = \gamma(r) (< 1).$$

Hence, function φ is well defined on the interval $(0, \frac{1}{L_0})$. Moreover, define functions g_3 and g_4 on the interval $(0, \frac{1}{L_0})$ by

$$(2.4) \quad g_3(r) = \frac{1}{2} g_2(r) + \varphi(r)$$

and

$$g_4(r) = \frac{1}{2(1 - L_0r)} [L + 2|\alpha| M g_3(r)] r.$$

Evidently $g_4(r) \in [0, 1)$, if function

$$p_4(r) = (L + 2|\alpha|Mg_3(r))r - 2(1 - L_0r) < 0$$

for each $r \in (0, \frac{1}{L_0})$. We have that $p_4(0) = -2 < 0$ and

$$p_4((\frac{1}{L_0})^-) = \frac{1}{L_0}(L + 2|\alpha|Mg_3(\frac{1}{L_0})^-) > 0,$$

since $g_3((\frac{1}{L_0})^-) > 0$. It follows that p_4 (i.e g_4) has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r^* the smallest such zero. Then, we have that

$$0 < g_4(r) < 1 \text{ for each } r \in (0, r^*).$$

It follows from the definition of function p_4 and r_A that

$$\begin{aligned} p_4(r_A) &= (L + 2|\alpha|Mg_3(r_A))r_A - 2(1 - L_0r_A) \\ &= Lr_A - 2(1 - L_0r_A) + 2|\alpha|Mg_3(r_A)r_A \\ &= 2|\alpha|Mg_3(r_A)r_A > 0, \end{aligned}$$

since $Lr_A - 2(1 - L_0r_A) = 0$ by (2.1). That is, we have that $r^* < r_A$. Hence, we deduce that

$$(2.5) \quad 0 < g_1(r) < 1$$

$$(2.6) \quad 0 < g_2(r)$$

$$(2.7) \quad 0 < g_3(r)$$

and

$$(2.8) \quad 0 < g_4(r) < 1 \text{ for each } r \in (0, r^*).$$

Next, we present the local convergence analysis of method (1.2).

THEOREM 2.1. *Let $F : D \subseteq X \rightarrow Y$ be a twice Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0, M > 0, N > 0, \alpha \in (-\infty, +\infty)$ $\{c_k\} \in \mathbb{R}$ such that (2.2) and the following conditions hold for each $x \in D$*

$$(2.9) \quad F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X),$$

$$(2.10) \quad \|F'(x^*)^{-1}(F(x) - F(x^*))\| \leq L_0\|x - x^*\|,$$

$$(2.11) \quad \|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*))\| \leq \frac{L}{2}\|x - x^*\|^2,$$

$$(2.12) \quad \|F'(x^*)^{-1}F'(x)\| \leq M,$$

$$(2.13) \quad \|F'(x^*)^{-1}F''(x)\| \leq N$$

and

$$(2.14) \quad \bar{U}(x^*, r^*) \subseteq D,$$

where r^* is defined above Theorem 2.1. Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \dots$ and converges to x^* . Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$,

$$(2.15) \quad \|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*,$$

$$(2.16) \quad \|K_n\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\|,$$

$$(2.17) \quad \|H_n\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\|$$

and

$$(2.18) \quad \|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|,$$

where the “ g ” functions are defined above Theorem 2.1. Furthermore, suppose that there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. Using (2.10), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*)$ we get that

$$(2.19) \quad \|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0r^* < 1.$$

It follows from (2.19) and the Banach Lemma on invertible operators [2, 11] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$(2.20) \quad \|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0r^*}.$$

Hence, y_0 and x_1 are well defined. Using the first substep in method (1.2) for $n = 0$, (2.11), (2.21), (2.5) and definition of r^* , we have in turn that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)] \end{aligned}$$

so,

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.15) for $n = 0$ and $y_0 \in U(x^*, r)$. It follows from the definition of $K_0, g_2, (2.12), (2.13)$ and (2.21) that

$$\begin{aligned} \|K_0\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \|F'(x^*)^{-1}F''(x_0)\| \\ &\quad \times \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*)) (x_0 - x^*) dt \right\| \\ &\leq \frac{MN\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\|, \end{aligned}$$

which shows (2.16) for $n = 0$. Moreover, using the definition of $H_0, h_2, \varphi, (2.16), g_3$, we get that

$$\begin{aligned} \|H_0\| &\leq \frac{1}{2}\|K_0\| + \sum_{k \geq 2} |c_k| \|K_0\|^k \\ &\leq \frac{1}{2}g_2(\|x_0 - x^*\|)\|x_0 - x^*\| + \sum_{k \geq 2} |c_k| g_2^k(\|x_0 - x^*\|)\|x_0 - x^*\|^k \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\|, \end{aligned}$$

which shows (2.17) for $n = 0$. Furthermore, from the second substep of method (1.2) for $n = 0, (2.15), (2.17), (2.12), (2.8)$ the definition of g_4 and r^* we get that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|y_0 - x^*\| + |\alpha| \|H_0\| \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times \left\| \int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*)) (x_0 - x^*) dt \right\| \\ &\leq \left[\frac{L}{2(1 - L_0\|x_0 - x^*\|)} + \frac{M|\alpha|g_3(\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|} \right] \|x_0 - x^*\|^2 \\ &\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.18) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing y_0, x_1 by y_m, x_{m+1} in the preceding estimates we arrive at estimates (2.15)-(2.18). Finally, from the estimates $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ we obtain $\lim_{k \rightarrow \infty} x_k = x^*$.

To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.10) we get that

$$\begin{aligned} |F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ (2.21) \qquad \qquad \qquad &\leq \int_0^1 (1 - \theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}R < 1. \end{aligned}$$

It follows from (2.11) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. \square

REMARK 2.2. 1. In view of (2.10) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.12) can be dropped and M can be replaced by

$$M(r) = 1 + L_0r.$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [2, 3] of the form

$$F'(x) = T(F(x))$$

where T is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^* . For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$. Moreover, (2.11) can be replaced by the popular stronger conditions

$$(2.22) \quad \|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\| \text{ for each } x, y \in D$$

or

$$\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)) - F'(x))\| \leq L(1 - t)\|x - x^*\|$$

for each $x, y \in D$ and $t \in [0, 1]$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2, 3].

4. The parameter r_A given in (2.4) was shown by us to be the convergence radius of Newton's method [2, 3, 11]

$$(2.23) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \dots$$

under the conditions (2.10) and (2.2). Since, $r^* < r_A$ the convergence radius r of the method (2.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.23).

As already noted in [2, 3] r_A is at least as large as the convergence ball given by Rheinboldt [12]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \rightarrow \frac{1}{3} \text{ as } \frac{L_0}{L} \rightarrow 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [13].

5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger (\mathcal{C}) conditions used in [10]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln \left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \right) / \ln \left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|} \right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln \left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [10] involving condition (\mathcal{C}_5) .

3. Numerical examples

We present numerical examples in this section.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}^2, D = \bar{U}(0, 1), x^* = (0, 0)^T$ and define function F on D by

$$(3.1) \quad F(x) = (\sin x, \frac{1}{3}(e^x + 2x - 1))^T.$$

Then, using (2.9)-(2.13), we get $L_0 = L = N = 1, M = \frac{1}{3}(e + 2)$. The parameters are given in Table 1.

EXAMPLE 3.2. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0, 1)$. Define F on D for $v = (x, y, z)^T$ by

$$(3.2) \quad F(v) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z)^T.$$

parameters	$\alpha = 1$	$\alpha = 0.01$
r_A	0.6667	0.6667
c_k	$(\frac{1}{5.6619})^k$	$(\frac{1}{5.6619})^k$
r^*	0.1512	0.3392
ξ_1	2.9246	1.9748

TABLE 1. Parameters of methods (1.2) for $\alpha = 1$ and $\alpha = 0.01$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = e$, $M = N = e$. The parameters are given in Table 2.

parameters	$\alpha = 1$	$\alpha = 0.01$
r_A	0.3249	0.3249
c_k	$(\frac{1}{10.3206})^k$	$(\frac{1}{10.3206})^k$
r^*	0.0368	0.0877
ξ_1	1.9909	1.9247

TABLE 2. Parameters of methods (1.2) for $\alpha = 1$ and $\alpha = 0.01$

EXAMPLE 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 101.5578008 = N$. The parameters are given in Table 2.

parameters	$\alpha = 1$	$\alpha = 0.01$
r_A	0.0045	0.0045
c_k	$(\frac{1}{168.77})^k$	$(\frac{1}{168.77})^k$
r^*	0.0007	0.0031
ξ_1	0.9999	1.0000

TABLE 3. Parameters of methods (1.2) for $\alpha = 1$ and $\alpha = 0.01$

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